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SUMS OF DISTANCES IN NORMED SPACES

by

Mostafa Ghandehari

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22a NAME OF RESPONSIBLE INDIVIDUAL: Mostafa Ghandehari

22b TELEPHONE (Include Area Code): (408) 646-2124

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SUMS OF DISTANCES IN NORMED SPACES

Mostafa Ghandehari

Department of Mathematics

Naval Postgraduate School

Monterey, California 93943

ABSTRACT

A geometric proof for the following theorem due to Martelli and Busenberg is given. Integral geometry is used to discuss special cases and related results.

Theorem. Let $x_1, \dots, x_r$ be r points on the unit sphere S of a normed space. Assume that the convex hull of $x_1, \dots, x_r$ is at distance d from the origin measured with respect to the norm. Then

$$\sum_{i<j} ||x_i - x_j|| \geq 2(r-1)(1-d).$$

Let X be a real normed linear space. For each finite subset $\{x_1, \dots, x_r\} \subset X$ let $s = s(x_1, \dots, x_r)$ denote the sum of all distances determined by pairs from $\{x_1, \dots, x_r\}$. That is, let

$$s(x_1, \dots, x_r) = \sum ||x_i - x_j||, \tag{1}$$

where the sum is taken over all integers, i, j, satisfying $1 \leq i < j \leq r$. Let $S = \{x : ||x|| = 1\}$ be the unit sphere of X.

Martelli and Busenberg [8] use inequalities in connection with work on autonomous systems of differential equations to prove the following theorem.

Theorem 1. Let $x_1, \ldots, x_r$ be r points on the unit sphere S of a normed space. Assume that the convex hull of $x_1, \ldots, x_r$ is at distance d from the origin measured with respect to the norm. Then

$$s(x_1, \ldots, x_r) \geq 2(r-1)(1-d). \tag{2}$$

To prove Theorem 1 we use the following theorem which was conjectured by Grünbaum and proved in [1].

Theorem 2. Let $x_1, \ldots, x_r$ be points in a real normed linear space X. Suppose p belongs to the convex hull of $\{x_1, \ldots, x_r\}$. Then

$$s(x_1, \ldots, x_r) \geq (2r-2) \min ||x_i - p||, \tag{3}$$

where the minimum is taken over all i satisfying $1 \leq i \leq r$.

Proof of Theorem 1. There is a point p with distance d from the origin which belongs to the convex hull of $\{x_1, \ldots, x_r\}$. There is an integer j, $1 \leq j \leq r$, such that $\min ||x_i - p|| = ||x_j - p||$. By Theorem 2 and the triangle inequality

$$s(x_1, \ldots, x_r) \geq 2(r-1) \min_i ||x_i - p|| = 2(r-1)||x_j - p|| \geq 2(r-1)(1-d),$$

where the last inequality is obtained by applying the triangle inequality to a triangle with vertices p, x_j and the origin. Thus the proof of Theorem 1 is completed. ■

In the following we review results related to the inequality (2). Consider r points $x_1, x_2, \ldots, x_r$ in a real normed linear space X with norm $||\cdot||$. The convex hull of midpoints of line segments joining x_i and x_j for all i and j, $i \neq j$, is called the *midpoint polyhedron* for $x_1, \ldots, x_r$. Chakerian and the author [3] proved the following.

Theorem 3. Let p belong to the midpoint polyhedron of $\{x_1, \ldots, x_r\} \subset X$. Then

$$(2r-2) \sum_{i=1}^{r} ||p - x_i|| \leq rs(x_1, \ldots, x_r). \tag{4}$$

As a consequence of the above the following is shown in [3].

Theorem 4. Let $x_1, \ldots, x_r$ be points on the unit sphere S of a normed linear space X, and suppose that the origin o belongs to the convex hall of $\{x_1, \ldots, x_r\}$. Then

$$s(x_1, \ldots, x_r) \geq 2r - 2. \tag{5}$$

Theorem 4 is due to Chakerian and Klamkin [4], which they proved for Euclidean spaces and for the Minkowski plane. Wolfe [10] proved Theorem 3 using the concept of metric dependence.

Figures 1 and 2 give examples where equalities are attained in Theorems 3 and 4. In the remainder of this article we use techniques from integral geometry to prove special cases of Theorem 2 in two and three-dimensional Minkowski spaces. Minkowski spaces are simply finite dimensional normed linear spaces. Smoothness assumptions on the boundary of the unit disk E for a Minkowski plane will enable us to use Crofton's simplest formula from integral geometry to give a proof of (4) for three points $\{x_1, x_2, x_3\}$. If the unit ball for a 3-dimensional Minkowski space is a zonoid, then we use integral geometry to prove (4) for the case of four points x_1, x_2, x_3, and x_4 forming a simplex. A *zonoid* is a limit of sums of segments. Bolker [2] discusses equivalent conditions for a convex subset of R^n to be a zonoid.

Santaló [9] is a good reference for integral geometry in the Euclidean spaces. Given a curve C in the Euclidean plane, let L denote the length of C. Crofton's simplest formula is

$$\int\int n dp d\theta = 2L. \tag{6}$$

where the integral is taken over all lines intersecting C, the pair (p, θ) is the polar coordinate representation of the foot of perpendicular from the origin to the line, and n is the number of intersections of a line with coordinates (p, θ) with C. The differential element $dG = dp d\theta$ is the *integral geometric density for lines.*

Chakerian [5] treats integral geometry in the Minkowski plane. We sketch the definitions he uses to develop Crofton's simplest formula in the Minkowski plane. Assume the unit circle E is "sufficiently" differentiable and has positive finite curvature everywhere. Parameterize

E by twice its sectorial area ϕ, and write the equation of E as

$$t = t(\phi), \qquad 0 \leq \phi \leq 2\pi, \qquad ||t|| = ||t - 0|| = 1.$$

E is called the *indicatrix.* Define the *isoperimetrix* T by the parametric representation

$$n(\phi) = \frac{dt(\phi)}{d\phi}, \qquad 0 \leq \phi \leq 2\pi.$$

Define $\lambda(\phi)$ by $\frac{dn(\phi)}{d(\phi)} = -\lambda^{-1}(\phi)t(\phi)$. Then the density for lines in two- dimensional Minkowski spaces is defined as follows. Let $G = G(p, \phi)$ be parallel to the direction $t(\phi)$. The equation of G is

$$[t(\phi), x] = p,$$

where $[x, y] = x_1y_2 - x_2y_1$. Then the *density* dG *for lines* is

$$dG = \lambda^{-1}(\phi)dpd\phi.$$

It is then shown in Chakerian [5] that the simplest formula of Crofton holds:

$$\int ndG = 2\ell \tag{7}$$

where n is the number of intersections of a line G with a curve C, integration is taken over all lines intersecting C and ℓ in the Minkowskian length of C. We use Crofton's simplest formula to prove the following Corollary of Theorem 3. Recall that we defined the midpoint polyhedron of r points earlier. In the case of three points the midpoint polyhedron is called the *midpoint triangle.*

Corollary 1. Consider a point p in the midpoint triangle of a triangle with vertices x_1, x_2, and x_3. Then

$$\sum_{i=1}^{3} ||p - x_i|| \leq \frac{3}{4} \sum_{1 \leq i < j \leq 3} ||x_i - x_j||. \tag{8}$$

Integral geometric proof. Let $\mathcal{L}_i$, $i = 1, 2, 3$ be the line segment joining p to x_i. Let $\ell_i = ||p - x_i||$. Let μ_i be the measure of lines which intersect $\mathcal{L}_i$ only. Assume μ_{ij} is the measure of the lines which intersect $\mathcal{L}_i$ and $\mathcal{L}_j$ and let $\ell(T)$ denote the length of the triangle with vertices x_1, x_2, x_3. Then

$$\ell(T) = \mu_1 + \mu_2 + \mu_3 + \mu_{12} + \mu_{23} + \mu_{31} = \mu_1 + \mu_{12} + \mu_{13} + \mu_2 + \mu_{21} + \mu_{23} + (\mu_3 - \mu_{12}).$$

Hence

$$\ell(T) = 2\ell_1 + 2\ell_2 + (\mu_3 - \mu_{12}).$$

Similarly,

$$\ell(T) = 2\ell_2 + 2\ell_3 + (\mu_1 - \mu_{23}),$$

and

$$\ell(T) = 2\ell_1 + 2\ell_3 + (\mu_2 - \mu_{13}).$$

Adding the last three inequalities we obtain,

$$3\ell(T) = 4(\ell_1 + \ell_2 + \ell_3) + (\mu_3 - \mu_{12}) + (\mu_1 - \mu_{23}) + (\mu_2 - \mu_{13}) \geq 4(\ell_1 + \ell_2 + \ell_3)$$

since $(\mu_3 - \mu_{12}) \geq 0$, $(\mu_1 - \mu_{23}) \geq 0$, $(\mu_2 - \mu_{13}) \geq 0$. To prove, for example, that $\mu_1 \geq \mu_{23}$, we reflect $\mathcal{L}_1$ through p and notice that any line which intersects $\mathcal{L}_2$ and $\mathcal{L}_3$ will intersect the reflection of $\mathcal{L}_1$, but there are lines which intersect the reflection of $\mathcal{L}_1$ and miss $\mathcal{L}_2$ and $\mathcal{L}_3$. We are using the fact that the measure of the lines which intersect the reflection of $\mathcal{L}_1$ only have the same measure as the lines which intersect $\mathcal{L}_1$ only. Note that equality holds if and only if reflection of $\mathcal{L}_1$ will coincide with $\mathcal{L}_2$ and $\mathcal{L}_3$. ■

As a consequence of the above we obtain the following result of Laugwitz [7]:

Corollary 2. A triangle inscribed in the unit circle of a Minkowski plane and having the center as an interior point has perimeter greater than 4.

For curves in three dimensional Euclidean spaces, the integral geometric analogue of Crofton's simplest formula is

$$\int\int\int n(\theta, \phi, p) \sin\theta \; d\theta d\phi dp = \pi L \tag{9}$$

where $n(\theta, \phi, p)$ in the number of intersections of a plane of coordinates (θ, ϕ, p) with the curve C and integration is taken over all planes intersecting C. See Santaló [9]. For the case where the unit ball is a zonoid, Chakerian [6], Appendix, gives the analogue of (9) for a Minkowski space. With this in mind we sketch a proof of the following special case of Theorem 2 (see Figure 3).

Corollary 3. Consider a tetrahedron with vertices x_1, x_2, x_3, and x_4 in a three-dimensional Minkowski space. Let p be a point in the midpoint polyhedron. Then

$$\sum_{i=1}^{4} ||p - x_i|| \leq \frac{2}{3} \sum_{1 \leq i < j \leq 4} ||x_i - x_j||. \tag{10}$$

Proof. Denote the line segment joining p to x_i by $\mathcal{L}_i$ and let $\ell_i = ||p - x_i||$. Let μ_i be the measure of planes intersecting $\mathcal{L}_i$ only. Suppose μ_{ij} is the measure of planes intersecting $\mathcal{L}_i$ and $\mathcal{L}_j$ only and similarly define μ_{ij}. Then,

$$\begin{aligned}
2\ell_1 &= 2\mu_1 + 2\mu_{12} + 2\mu_{13} + 2\mu_{14} + 2\mu_{124} + 2\mu_{134} + 2\mu_{123}, \\
2\ell_2 &= 2\mu_2 + 2\mu_{21} + 2\mu_{23} + 2\mu_{24} + 2\mu_{213} + 2\mu_{214} + 2\mu_{234}, \\
2\ell_3 &= 2\mu_3 + 2\mu_{31} + 2\mu_{32} + 2\mu_{34} + 2\mu_{314} + 2\mu_{324} + 2\mu_{321}
\end{aligned}$$

The sum of the edge lengths of the tetrahedron is denoted by $L(T)$ and is given by

$$\begin{aligned}
\ell(T) &= 3\mu_1 + 3\mu_2 + 3\mu_3 + 3\mu_{123} + 3\mu_{124} + 3\mu_{134} + 3\mu_{234} \\
&\quad +4[\mu_{12} + \mu_{23} + \mu_{34} + \mu_{13} + \mu_{14} + \mu_{24}].
\end{aligned}$$

The expression in brackets is multiplied by 4 since any line intersecting $\mathcal{L}_i$ and $\mathcal{L}_j$ intersects the tetrahedron in 4 points. Hence,

$$\begin{aligned}
\ell(T) - 2(\ell_1 + \ell_2 + \ell_3) &= (\mu_1 - \mu_{234}) + (\mu_2 - \mu_{134}) + (\mu_3 - \mu_{124} \\
&\quad +3(\mu_4 - \mu_{123}) + 2(\mu_{34} + \mu_{14} + \mu_{24}).
\end{aligned}$$

But using reflection $(\mu_i - \mu_{jk\ell}) \geq 0$, $i \neq j, k, \ell$. Hence $\ell(T) \geq 2(\ell_1 + \ell_2 + \ell_3)$. Similarly $\ell(T) \geq 2(\ell_2 + \ell_3 + \ell_4)$, $\ell(T) \geq 2(\ell_1 + \ell_3 + \ell_4)$ and $\ell(T) \geq 2(\ell_1 + \ell_2 + \ell_4)$ which yields $4\ell(T) \geq 6(\ell_1 + \ell_2 + \ell_3 + \ell_4)$, proving (10). ■

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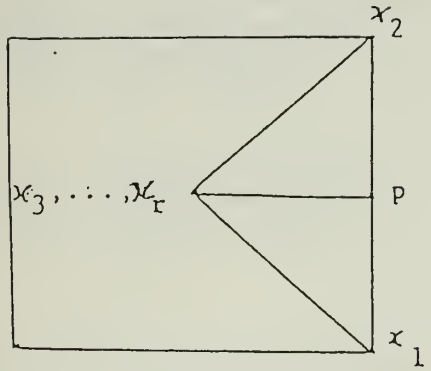

Figure 1 Equality for Theorem 3.

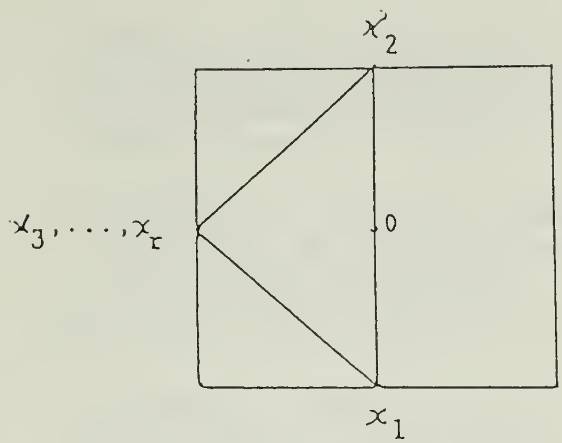

Figure 2 Equality for Theorem 4.

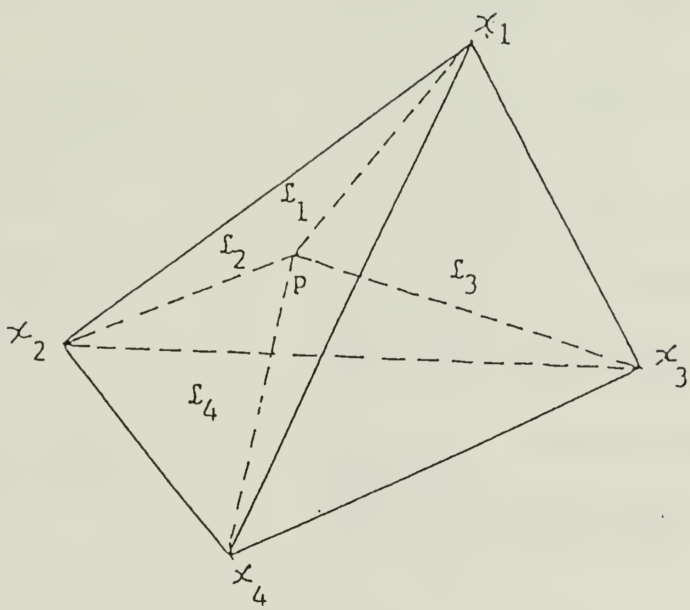

Figure 3 For inequality (10).

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